Stochastic Calculus: Ito's integral

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First Order Variation (FV)

Definition: First Order Variation of a function f

$$FV(f) = \lim_{n \to \infty} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| \quad (*)$$

- The partition is $0 = t_0 < t_1 < \cdots < t_n = T$.
- Taking $t_i = \frac{iT}{n}$.

First Order Variation (FV)

For a differentiable function f

Using the Mean Value Theorem, $(*)$ can be written as:

$$FV(f) = \lim_{n\to\infty} \sum_{i=0}^{n-1} |f'(t_i^*)|(t_{i+1} - t_i), \quad t_i \leq t_i^* \leq t_{i+1}$$

$$= \int_0^T |f'(t)| dt$$

Conclusion: For a continuously differentiable function, FV is finite.

First Order Variation of Brownian Motion (BM)

FV of Brownian Motion $\{B_t : 0 \leq t \leq T\}$

$$FV(B) = \lim_{n\to\infty} \sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)|$$

- The increment $\Delta B_i = B(t_{i+1}) - B(t_i) \sim N(0, t_{i+1} - t_i)$.
- Let $Z_i \sim_{i.i.d.} N(0, 1)$, then $\Delta B_i = \sqrt{t_{i+1} - t_i} Z_i$.

First Order Variation of Brownian Motion (BM)

Again, with $t_i = \frac{iT}{n}$, $t_{i+1} - t_i = \frac{T}{n}$, in distribution

$$FV(B) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\frac{T}{n}} |Z_i|$$
$$= \lim_{n\to\infty} \sqrt{nT} \left(\frac{1}{n} \sum_{i=0}^{n-1} |Z_i| \right)$$

$\frac{1}{n} \sum_{i=0}^{n-1} |Z_i| \to E[|Z_1|] = \sqrt{2/\pi}$ by SLLN.

The expression inside the parenthesis behaves like $\sqrt{n} \cdot (\text{constant})$.

Thus, the limit, the first order variation of BM, is infinite

Quadratic Variation (QV)

Definition: Quadratic Variation of a function f

$$QV(f) = \lim_{n\to\infty} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^2$$

- For a continuous function f:

$$QV(f) \le \max_{0\le i\le n-1} |f(t_{i+1}) - f(t_i)| \times \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$$

$$\to 0 \times FV(f) \quad \text{if } FV(f) < \infty$$

  Since f is continuous, $\max_{0\le i\le n-1} |f(t_{i+1}) - f(t_i)| \to 0$.

- If f has finite FV then $QV(f) = 0$.

Quadratic Variation of Brownian Motion

$$QV(B_T) = \lim_{n\to\infty} \sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)|^2$$

- Again, $B(t_{i+1}) - B(t_i)$ equals $\sqrt{T/n}\, Z_i$ in distribution (Z_i is $N(0,1)$).
- Heuristically,

$$QV(B_T) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{T}{n} Z_i^2 = \lim_{n\to\infty} T \left(\frac{1}{n} \sum_{i=0}^{n-1} Z_i^2 \right)$$

- Now, $E[Z_i^2] = 1$. By the Strong Law of Large Numbers (SLLN):

$$\frac{1}{n} \sum_{i=0}^{n-1} Z_i^2 \to E[Z_1^2] = 1$$

- Thus, $QV(B_T) = T$.

Quadratic Variation of BM (more precise): L^2 Convergence

Convergence in L^2

- Let $X_n = \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^2$.
- We want to show $X_n \xrightarrow{L^2} T$, which means $E[|X_n - T|^2] \to 0$.
- We know $E[X_n] = T$ (because $E[(B_{t_{i+1}} - B_{t_i})^2] = t_{i+1} - t_i$).
- Need to show $\mathbf{Var(X_n)} \to \mathbf{0}$ as $n \to \infty$.

L_2 Convergence

Variance Calculation

Using $B_{t_{i+1}} - B_{t_i} = \sqrt{\frac{T}{n}} Z_i$, so $(B_{t_{i+1}} - B_{t_i})^2 = \frac{T}{n} Z_i^2$.

$$
\begin{aligned}
Var(X_n) &= \sum_{i=0}^{n-1} Var\left(\frac{T}{n} Z_i^2\right) \quad \text{(due to independence)} \\
&= \sum_{i=0}^{n-1} \frac{T^2}{n^2} Var(Z_i^2) = n \cdot \frac{T^2}{n^2} Var(Z_1^2) \\
&= \frac{T^2}{n} Var(Z_1^2)
\end{aligned}
$$

Since $Var(Z_1^2)$ is a finite constant (it's 2), $\lim_{n\to\infty} Var(X_n) = 0$.

Itô's Integral $\int_0^T f \, dB_t$

- Used to define the value of trading strategies driven by change in underlying assets.
- Defined for "nice functions" $f(\omega, s)$ (a stochastic process) measurable w.r.t. product measure $dP \times dt$,
    - a measure on the smallest sigma algebra containing sets $A \times [t_1, t_2]$ where $A \in \mathcal{F}$ and $0 \leq t_1 \leq t_2 \leq T$, and

$$(d\mathbb{P} \times dt)(A \times [t_1, t_2]) = \mathbb{P}(A) \times (t_2 - t_1).$$

- We require $E\left[\int_0^T f^2(\omega, s) ds\right] < \infty$ and $f(\omega, s)$ must be $\mathcal{F}_s$-measurable $\forall s$ (adapted).

Defining Itô's Integral

- $\mathcal{L}^{\mathbf{2}}{}_{(d\mathbb{P}\times dt)}$: The class of all functions f satisfying $E\left[\int_0^T f^2 ds\right] < \infty$.
- $\mathcal{H}^2$: $f \in \mathcal{L}^{\mathbf{2}}{}_{(d\mathbb{P}\times dt)}$ and $f(\omega, s)$ must be $\mathcal{F}_s$-measurable $\forall s$
- $\mathcal{H}_{\mathbf{0}}^{\mathbf{2}}$: The class of **simple functions** f defined as:

$$f(\omega, t) = \sum_{i=0}^{n-1} a_i(\omega) I_{(t_i < t \leq t_{i+1})}$$

where $a_i(\omega) \in \mathcal{F}_{t_i}$ and $E[a_i^2(\omega)] < \infty$.

Defining Itô's Integral

Itô's Integral for Simple Functions

For $f \in \mathcal{H}_0^2$:

$$\int_0^T f(\omega, t)dB_t = \sum_{i=0}^{n-1} a_i(\omega)(B_{t_{i+1}} - B_{t_i})$$

- This is the analog of simple functions in the Lebesgue integral definition.
- $\mathcal{H}_0^2 \subset \mathcal{L}^{\mathbf{2}}{}_{(d\mathbb{P}\times dt)}$ (space of all rv with finite second moment) is easy to verify.

Big Claim: Any function $f \in \mathcal{H}^2$ can be approximated by a function in $\mathcal{H}_0^2$.

Itô's Geometry (Isometry)

The Isometry Property

For $f \in \mathcal{H}^2$,

$$E\left[\left(\int_0^T f(\omega,t)dB_t\right)^2\right] = E\left[\int_0^T f^2(\omega,t)dt\right]$$

- Equivalently,

$$\|I_T(f)\|_{\mathcal{L}^2(d\mathbb{P})} = \|f\|_{\mathcal{L}^2(d\mathbb{P}\times dt)}$$

where $I_T(f) = \int_0^T f(\omega,t)dB_t$.

Proof Sketch for Simple Functions $f \in \mathcal{H}_0^2$

$$E\left[\left(\sum_{i=0}^{n-1} a_i(B_{t_{i+1}} - B_{t_i})\right)^2\right] = E\left[\sum_{i=0}^{n-1} a_i^2(B_{t_{i+1}} - B_{t_i})^2\right] + \sum_{j \neq k} E[\ldots]$$

- **Cross Terms Vanish:** Since $B_{t_{k+1}} - B_{t_k}$ is independent of $\mathcal{F}_{t_k}$ (and $a_j, a_k, B_{t_{j+1}} - B_{t_j}$ for $j < k$), the cross terms vanish:

$$E\left[E[a_j a_k \Delta B_j \Delta B_k \mid \mathcal{F}_{\max(t_j, t_k)}]\right] = 0$$

- **Diagonal Terms:**

$$\begin{aligned} E\left[\sum a_i^2(B_{t_{i+1}} - B_{t_i})^2\right] &= E\left[E\left[\sum a_i^2(B_{t_{i+1}} - B_{t_i})^2 \mid \mathcal{F}_{t_i}\right]\right] \\ &= E\left[\sum a_i^2(t_{i+1} - t_i)\right] \end{aligned}$$

This equals $E\left[\int_0^T f^2(\omega, s)ds\right]$.

Generalizing Itô's Integral

- Result: Given any $f \in \mathcal{H}^2$, there exist Let $\{f_n\}_{n\in\mathbb{N}}$, a sequence of simple functions in $\mathcal{H}_0^2$, such that

$$f_n \xrightarrow{\mathcal{L}^2(d\mathbb{P}\times dt)} f.$$

- Now $\mathcal{L}^2(d\mathbb{P} \times dt)$ is a complete space (any $\mathcal{L}^2$ space is complete. Not proved).
- The convergence of f_n to f implies $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^2(d\mathbb{P} \times dt)$.
- By the Itô Isometry:

$$\left\| \int_0^T (f_n - f_m) dB_t \right\|_{\mathcal{L}^2(d\mathbb{P})} = \|f_n - f_m\|_{\mathcal{L}^2(d\mathbb{P}\times dt)}$$

Existence of general Itô's Integral

- Since $\|f_n - f_m\|_{\mathcal{L}^2(d\mathbb{P}\times dt)} \to 0$, the sequence of integrals $\left\{\int_0^T f_n dB_t\right\}$ is a Cauchy sequence in $\mathcal{L}^2(d\mathbb{P})$.
- $\mathcal{L}^2(d\mathbb{P})$ is also complete.
- Thus, the limit $\lim_{n\to\infty}\int_0^T f_n dB_t$ exists in $\mathcal{L}^\in(\lceil\mathbb{P})$.

Definition: $\int_0^T f(\omega, t) dB_t$ is defined as this limit.

- Let $I_t(f) = \int_0^t f(\omega, s) dB_s$ for $f \in \mathcal{H}_0^2$, where it takes value $a_i(\omega) \in \mathcal{F}_{t_i}$ for $t \in (t_i, t_{i+1}]$ where $0 = t_0 < \ldots < t_m = T$.

- Then, for $t \in (t_k, t_{k+1}]$

$$\int_0^t f(\omega, s) dB_s = \sum_{i=0}^{k-1} a_i(\omega)(B_{t_{i+1}} - B_{t_i}) + a_k(\omega)(B_t - B_{t_k})$$

- Easy to check that

$$\mathbf{E}[\mathbf{I_t}(\mathbf{f}) \mid \mathcal{F}_\mathbf{u}] = \mathbf{I_u}(\mathbf{f}) \quad \text{for } u < t$$

- This is shown by splitting the integral at u and using the $\mathcal{F}_s$-measurability of a_s and the independent increments of BM. (Show!)

Martingale property of Itô's Integral

- The above holds for general $f \in \mathcal{H}^2$.
- **Consequence:** $I_t(f)$ is a martingale, so $E[I_t(f)] = E[I_0(f)] = 0$.
- Can show **Quadratic Variation:**

$$QV\left(\int_0^t f(\omega, s)dB_s\right) = \int_0^t f^2(\omega, s)ds$$

Properties of Itô's Integral $I_t(f) = \int_0^t f(\omega, s) dB_s$

1. $I_t(f)$ is a **continuous** function of t, $\forall \omega$.
2. $I_t(f)$ is $\mathcal{F}_t$-**measurable** $\forall t$.
3. $I_t(f)$ is **linear**.
4. $I_t(f)$ is a **martingale**.
5. Itô's Isometry holds.
6. $QV(I_t(f)) = \int_0^t f^2(\omega, s) ds$.

Example: $\int_0^t B_s dB_s$

Mean and Variance

Let $I_t(B) = \int_0^t B_s dB_s$. Since $f(s) = B_s \in \mathcal{H}^2$, $I_t(B)$ is a martingale.

- **Mean:** $E[I_t(B)] = 0$.
- **Variance (using Itô Isometry):**

$$
\begin{aligned}
V(I_t(B)) &= E\left[\left(\int_0^t B_s dB_s\right)^2\right] \\
&= E\left[\int_0^t B_s^2 ds\right] \quad \text{(Isometry)} \\
&= \int_0^t E[B_s^2] ds \quad \text{(Fubini)} \\
&= \int_0^t s ds = \frac{t^2}{2}.
\end{aligned}
$$

Example: Closed Form for $\int_0^t B_s dB_s$

1. Approximate B_s by simple functions $B_s^{(n)} = \sum_{i=0}^{n-1} B_{s_i} I_{(s_i < s \leq s_{i+1})}$.

2. Show $B_s^{(n)} \to B_s$ in $\mathcal{L}^2_{d\mathbb{P}\times ds}$:

$$E\left[\int_0^t (B_s^{(n)} - B_s)^2 ds\right] = \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} E[(B_{s_i} - B_s)^2] ds$$

3. Since $E[(B_{s_i} - B_s)^2] = |s - s_i|$, the integral is:

$$\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} (s - s_i) ds = \sum_{i=0}^{n-1} \frac{(s_{i+1} - s_i)^2}{2} \to 0 \quad \text{as } n \to \infty$$

Integral as a limit

$$\int_0^t B_s dB_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} B_{s_i}(B_{s_{i+1}} - B_{s_i})$$
$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{B_{s_{i+1}}^2 - B_{s_i}^2}{2} - \frac{1}{2}(B_{s_{i+1}} - B_{s_i})^2 \right)$$

since $a(b-a) = \frac{b^2-a^2}{2} - \frac{(b-a)^2}{2}$. Now,

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{B_{s_{i+1}}^2 - B_{s_i}^2}{2} \right) = \frac{B_t^2}{2} \quad \text{(Telescopic Sum)}$$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{1}{2}(B_{s_{i+1}} - B_{s_i})^2 \right) = \frac{1}{2} QV(B_t) = \frac{t}{2}$$

$$\implies \int_0^t \mathbf{B_s dB_s} = \frac{\mathbf{B_t^2}}{\mathbf{2}} - \frac{\mathbf{t}}{\mathbf{2}}$$

Itô's Formula (for $f(B_t)$)

For a twice continuously differentiable function f:

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds$$

Derivation Sketch (Taylor Series)

$$f(B_t) - f(0) = \sum_{i=0}^{n-1}(f(B_{t_{i+1}}) - f(B_{t_i}))$$

The Taylor series approximation (up to second order):

$$f(B_{i+1}) - f(B_i) \approx f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2}f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2$$

Itô's Formula (for $f(B_t)$)

- The first term converges to the Itô Integral:

$$\sum_{i=0}^{n-1} f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) \to \int_0^t f'(B_s)dB_s$$

- The second term may be re-expressed as:

$$\frac{1}{2}\sum_{i=0}^{n-1} f''(B_{t_i})\left((B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\right) + \frac{1}{2}\sum_{i=0}^{n-1} f''(B_{t_i})(t_{i+1} - t_i)$$

The first term converges to zero, and the second to

$$\frac{1}{2}\int_0^t f''(B_s)ds$$

Itô's Formula for $f(t,x)$

Observing that

$\lim_{n\to\infty} \sum_{i=0}^{n} (t_{i+1} - t_i)^2 \quad = \lim_{n\to\infty} \frac{t^2}{n^2} \cdot n = 0$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)(B_{t_{i+1}} - B_{t_i}) = 0.$$

Ito's lemma extends as

$$f(t, B_t) = f(0,0) + \int_0^t f_t(s, B_s)ds + \int_0^t f_x(s, B_s)dB_s + \frac{1}{2}\int_0^t f_{xx}(s, B_s)ds$$

where subscript t denotes partial derivative w.r.t. first argument, x denotes partial derivative and xx denotes double derivative w.r.t. second argument.

In short form

$$df(t, B_t) = f_t(t, B_t)dt + f_x(t, B_t)dB_t + \frac{1}{2} f_{xx}(t, B_t)dt$$

Example

- Consider $f(t, B_t) = \exp(\theta B_t - \theta^2 t/2)$
- Then $f_t(t,x) = -\frac{\theta^2}{2} f(t,x)$, $f_x(t,x) = \theta f(t,x)$ and $f_{xx}(t,x) = \theta^2 f(t,x)$.
- By Ito's lemma

$$f(t, B_t) = f_t(t, B_t)dt + f_x(t, B_t)dB_t + \frac{1}{2} f_{xx}(t, B_t)dt$$

so that

$$\exp(\theta B_t - \theta^2 t/2) = 1 + \theta \int_0^t \exp(\theta B_s - \theta^2 s/2) dB_s$$

and is a martingale.

Itô's Process

Itô's Process

A process $(\mathbf{X_t})_{\mathbf{t \geq 0}}$ is an **Itô's process** if it is defined by:

$$dX_t = \mu_t dt + \sigma_t dB_t$$

Actual Meaning (Integral Form):

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s$$

- where drift process $(\mu_t : t \leq T)$ and diffusion process $(\sigma_t : t \leq T)$ are adapted - μ_s and σ_s are $\mathcal{F}_s$-measurable for each s.

General Itô's Formula (for $f(t, X_t)$)

Itô's Formula for $f(t, X_t)$

Let $f(t,x)$ be a function that is continuous in t and twice continuously differentiable in x, and let X_t be an Itô process: $dX_t = \mu_t dt + \sigma_t dB_t$. The new representation for $f(t, X_t)$ obtained refining the telescopic expansion has the form :

$$df(t, X_t) = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} dX_t^2$$

where $dX_t^2 = (\mu_t dt + \sigma_t dB_t)^2$:

$$dX_t^2 = \mu_t^2 (dt)^2 + 2\sigma_t \mu_t (dt dB_t) + \sigma_t^2 (dB_t)^2$$

Rules for second order terms

- $dB_t \cdot dB_t = dt$ (Quadratic Variation of BM)
- $dt \cdot dt = \lim_{n\to\infty} \sum_{i=0}^{n} (t_{i+1} - t_i)^2 = 0$
- $dt \cdot dB_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)(B_{t_{i+1}} - B_{t_i}) = 0.$
- It follows that

$$\mathbf{dX_t^2 = \sigma_t^2 dt}$$

General Itô's Formula (for $f(t, X_t)$)

Thus,

$$df(t, X_t) = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} \sigma_t^2 dt$$

Integral Form:

$$f(t, X_t) = f(0, X_0) + \int_0^t \left(f_t + f_x \mu_s + \frac{1}{2} f_{xx} \sigma_s^2 \right) ds + \int_0^t f_x \sigma_s dB_s$$

Example, Ito process

- Consider an asset price process

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t \tag{1}$$

where $\{\mu_t S_t\}$ and $\{\sigma_t S_t\}$ are adapted processes.

- Let $Y_t = \log S_t$. Applying Itô's Lemma ($f(S_t) = \log S_t$).

$$dY_t = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}\sigma_t^2 S_t^2 dt = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dB_t$$

$$\Rightarrow \ln S_t = \ln S_0 + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dB_s$$

- Therefore (1) is equivalent to

$$S_t = S_0 e^{\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dB_s}$$

Example, deterministic integrand

- Consider $X_t = \int_0^t f(s)dB_s$. What is the distribution of X_t? (here $dX_t = f(t)dB_t$)

- Let $U_t = \exp(\theta \int_0^t f(s)dB_s - \frac{\theta^2}{2}\int_0^t f(s)^2 ds)$.

- Let
$$W_t = \theta \int_0^t f(s)dB_s - \frac{\theta^2}{2}\int_0^t f(s)^2 ds$$
denote an Ito process. $U_t = e^{W_t}$.

- By Ito's lemma
$$dU_t = U_t dW_t + \frac{1}{2}U_t(dW_t)^2.$$

- It follows that

$$dU_t = \theta U_t f(t) dB_t - \frac{\theta^2}{2} U_t f(t)^2 dt + \frac{\theta^2}{2} U_t f(t)^2 dt = \theta U_t f(t) dB_t.$$

- Hence, U_t is a martingale and $EU_t = EU_0 = 1$.

- Therefore

$$\exp\left(\theta \int_0^t f(s) dB_s\right) = \exp\left(\frac{\theta^2}{2} \int_0^t f(s)^2 ds\right).$$

What is the distribution of $\int_0^t f(s) dB_s$?